S. S. College. Jehanabad (Magadh University)

Department : Physics

Subject : Quantum Mechanics

Class : B.Sc(H) Physics Part III

Topic: Angular Momentum in Quantum Mechanics

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Orbital Angular Momentum in Quantum Mechanics

The angular momentum in classical mechanics is widely studied but in this note we will discussed angular momentum in Quantum mechanics. It is an intrinsic property of elementary particles which is also called spin angular momentum. It is not found in classical mechanics.

The total angular momentum of a particle is the vector sum of the orbital and spin angular momenta.

1. Orbital Angular Momentum operator

classically, the angular momentum $\vec{L}$

$$\vec{L} = \vec{r} \times \vec{p}$$

and its cartesian components is given by:

$$\begin{aligned} L_x &= y p_z - z p_y \\ L_y &= z p_x - x p_z \\ L_z &= x p_y - y p_x \end{aligned} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

The corresponding quantum mechanical operators are obtained by replacing $\vec{p}$ and its component with respective operators.

$$\vec{L} = -i\hbar (\vec{r} \times \vec{\nabla})$$

and

$$L_x = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right)$$

$$L_y = -i\hbar \left(z \partial/\partial x - x \partial/\partial z \right)$$

$$L_z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

Commutation Relations

$$[x, P_x] = [y, P_z] = [z, P_z] = i\hbar$$

By using this result, we can calculate

$$[L_x, L_y] = L_x L_y - L_y L_x$$

$$= [(yP_z - zP_y)(zP_x - xP_z) - (zP_x - xP_z)(yP_z - zP_y)]$$

$$= yP_z zP_x - yP_z xP_z - zP_y zP_x + zP_y xP_z - zP_x yP_z$$

$$= yP_x(P_z z - zP_z) + P_y x\,(zP_z - P_z z)$$

$$= (zP_z - P_z z)(xP_y - yP_x)$$

$$= [z, P_z](xP_y - yP_x)$$

$$= i\hbar L_z$$

In similar manner,

$$\left.\begin{aligned} [L_x, L_y] &= i\hbar L_z \\ [L_y, L_z] &= i\hbar L_x \\ [L_z, L_x] &= i\hbar L_y \end{aligned}\right] \text{(commutation Relation)}$$

Now

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

Evaluate:

$$[L^2, L_x] = [L_x^2 + L_y^2 + L_z^2, L_x]$$

as $[L^2_x, L_x] = 0$, $[L_x, L_x] = 0$

Then

$$[L^2, L_x] = [L_y^2, L_x] + [L_z^2, L_x]$$

$$= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z$$

$$= -i\hbar L_y L_z + (-i\hbar L_z L_y) + L_z L_y(i\hbar) + i\hbar L_y L_z$$

$$= -i\hbar(L_y L_z + L_z L_y) + i\hbar(L_z L_y + L_y L_z)$$

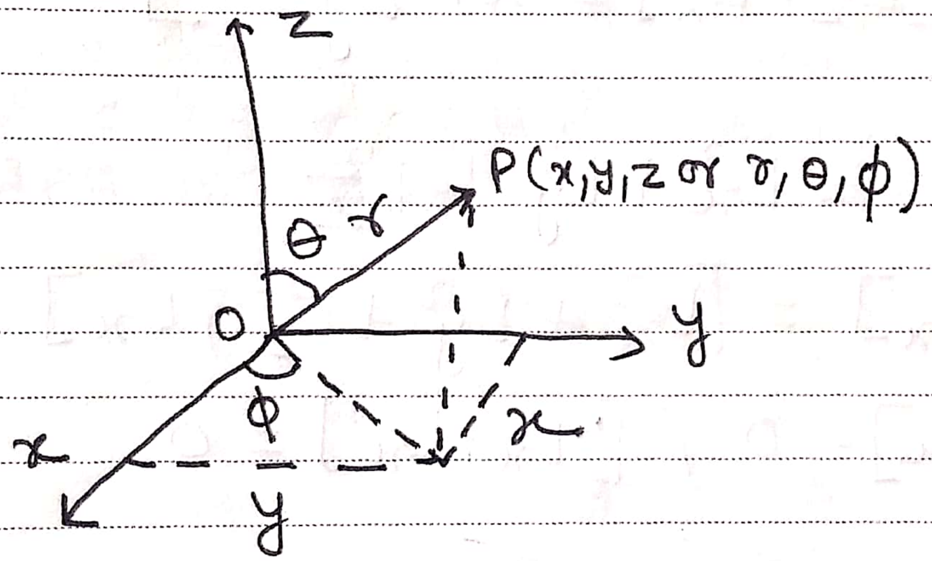
$$= 0$$

In similar way, we can show that

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$$

Therefore it is possible to find L^2 and L_x, L_y and L_z simultaneously.

Angular momentum operators in spherical coordinates



$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$0 \le r \le \infty$, $\theta \to 0$ to π, $\phi \to 0$ to 2π

Then

$$L_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

$$L_y = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

and

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \quad -(1)$$

Eigen values and Eigen functions of L^2 and L_z

Let us denote the eigen values of

$L^2 \to \lambda\hbar^2$

$L_z \to m_l\hbar$

and the corresponding common eigenfunction be $Y(\theta,\phi)$

Then

$$L^2 Y(\theta,\phi) = \lambda\hbar^2 Y(\theta,\phi) \quad -(2)$$

and

$$L_z Y(\theta,\phi) = m_l\hbar\, Y(\theta,\phi) \quad -(3)$$

from eq (1) & (2)

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}+\lambda Y=0 \qquad -④$$

Now the equation can be solved from method of separation of variables :

$$Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi) \qquad -⑤$$

from eq (4) & (5), we get and multiply eq(4) by $\frac{\sin^2\theta}{Y(\theta,\phi)}$

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \frac{\sin^2\theta}{\Theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}+\lambda\Theta\right)\right]$$

$$\Rightarrow -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \frac{\sin^2\theta}{\Theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}+\lambda\Theta\right)\right]$$

As variable is separated and we get

$$\frac{d^2\Phi}{d\phi^2}+m_l^2\Phi=0 \qquad — ⑥$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(\lambda-\frac{m_l^2}{\sin^2\theta}\right)\Theta=0 \qquad -⑦$$

from ⑥,

$$\Phi(\phi) = Ae^{im_l\phi}$$

where A is constant and

$$\Phi(\phi+2\pi) = \Phi(\phi)$$

$$e^{2\pi m_l i} = 1 \Rightarrow m_l = 0, \pm 1, \pm 2$$

and $A = \dfrac{1}{\sqrt{2\pi}}$,

$$\Phi_{m_l}(\phi) = \frac{1}{\sqrt{2\pi}} e^{i m_l \phi} \; ; \; m_l = 0, \pm 1, \pm 2, - - -$$